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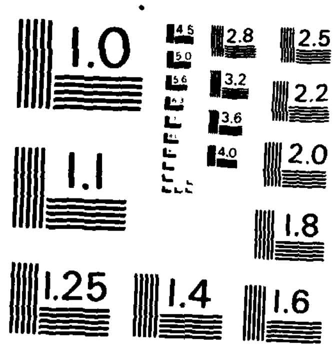

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ON A VARIATIONAL APPROACH TO SOME PARAMETER ESTIMATION PROBLEMS*

by

H.T. Banks

May 1985 LCDS #85-14

Lefschetz Center for Dynamical Systems

Division of Applied Mathematics Brown University Providence RI 02912

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ON A VARIATIONAL APPROACH TO SOME PARAMETER ESTIMATION PROBLEMS*

H.T. Banks+
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

May, 1985

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ABSTRACT

~~We~~ consider examples (1-D seismic, large flexible structures, bioturbation, nonlinear population dispersal) in which a variational setting can provide a convenient framework for convergence and stability arguments in parameter estimation problems. (Keywords: finite [illegible] partial differential equations [illegible])

I. Introduction

In this note we consider one aspect (arguments for convergence and stability via a variational approach) of least-squares formulations of parameter estimation problems for partial differential equations. Conceptually, one has a dynamical model with "states" $u = u(t,x)$, $0 \leq t \leq T$, $x \in \Omega$, and parameters $q = q(t,x)$ in some admissible set Q. Given observations or data, $z \in Z$, of some type (e.g., $z = \{\hat{u}_{ij}\}$ as observations $\{u(t_i,x_j)\}$), one wishes to determine parameters $\bar{q}$ that give a best fit of the model to the data. That is, one has the constrained optimization problem: From an admissible parameter set Q, choose a parameter $\bar{q}$ so that the corresponding solution of the dynamical model gives the best fit to data using a least-squares fit criterion.

Abstractly, we have a state space H in which we solve a dynamical system (S) for parameter dependent solutions $u = u(q)$ with the parameters chosen from some infinite dimensional set Q. If $C: H \to Z$ is a mapping from the state space to the observation space Z, the problem is one of minimizing

$$J(q,z) = \left|Cu(q) - z\right|_Z^2 \tag{1.1}$$

over $q \in Q$, where $|\cdot|_Z$ is an appropriately chosen norm in Z.

The fact that many problems of interest are infinite dimensional in both state spaces H and parameter sets Q leads to a rich class of mathematical questions including well-posedness, stability, and computational approaches. For example, consider the possibility of approximating the state space H by a sequence H^N of finite dimensional spaces and approximating the parameter set Q by a sequence Q^M of finite dimensional sets so as to obtain approximating problems: Minimize

$$J^N(q,z) = \left|Cu^N(q) - z\right|_Z^2 \tag{1.2}$$

over $q \in Q^M$, where u^N is an approximate solution to (S) lying in H^N. An important question concerns the ways in which Q^M approximating Q and H^N approximating H might guarantee convergence of solutions $\bar{q}^{N,M}$ of the problems of minimizing J^N over Q^M to a solution $\bar{q}$ of the problem of minimizing J over Q. A number of results [8], [27] in this area are available and we just sketch one set of arguments here (for examples and more details, see [8])

Suppose that the sets Q and Q^M lie in some metric space $\tilde{Q}$ and that, in fact, there is a mapping i^M: $Q \to Q^M$ so that $Q^M = i^M(Q)$. Further, assume that the following hypotheses are satisfied by Q^M and H^N:

(i) For <u>any</u> $q^k \to q$ in $\tilde{Q}$ we have $Cu^N(q^k) \to Cu(q)$ in Z as $N,k \to \infty$;

(ii) For each N, the mapping $q \to J^N(q,z)$ is continuous in the $\tilde{Q}$ topology;

(iii) The sets Q and Q^M, for each M, are compact in the $\tilde{Q}$ topology;

(iv) For each $q \in Q$, $i^M(q) \to q$ in $\tilde{Q}$ with the convergence uniform in $q \in Q$.

Under these assumptions, let $\bar{q}^{N,M}$ be solutions for the problems for (1.2) and let $\hat{q}^{N,M} \in Q$ be such that $i^M(\hat{q}^{N,M}) = \bar{q}^{N,M}$. From the compactness of Q, we may select subsequences, again denoted by $\{\hat{q}^{N,M}\}$ and $\{\bar{q}^{N,M}\}$ so that $\hat{q}^{N,M} \to \bar{q} \in Q$ and $\bar{q}^{N,M} \to \bar{q}$ (the latter follows from (iv)). The optimality of $\bar{q}^{N,M}$ guarantees that for every $q \in Q$

$$J^N(\bar{q}^{N,M},z) \leq J^N(i^M(q),z). \tag{1.3}$$

Using (i) and (iv) and taking the limit as $N,M \to \infty$ in this inequality yields $J(\bar{q},z) \leq J(q,z)$ for every $q \in Q$, or that $\bar{q}$ is a solution of the problem for (1.1). (Under uniqueness assumptions on the problems, one can actually guarantee convergence of the entire sequence $\bar{q}^{N,M}$ in place of subsequential convergence to solutions.)

We note that the essential aspects in the arguments sketched here involve compactness assumptions on the sets Q^M and Q. Such compactness ideas play a fundamental role in other theoretical and computational aspects of these problems. For example, one can formulate distinct concepts of <u>problem stability</u> and <u>method stability</u> involving some type of continuous dependence of solutions on the observations z in Z, and use hypotheses similar to (i) - (iv), with compactness again playing a critical role, to guarantee stability. We illustrate with a simple form of <u>method stability</u> (other stronger forms are also amenable to this approach).

We might say that an <u>approximation method</u>, such as that formulated above involving Q^M, H^N and (1.2), is <u>stable</u> if

$$\text{dist}\,(\tilde{q}^{N,M}(z^k),\tilde{q}(z^0)) \to 0$$

as $N,M,k \to 0$ for any $z^k \to z^0$ in Z, where $\tilde{q}(z)$ denotes the set of all solutions of the problem for (1.1) and $\tilde{q}^{N,M}(z)$ denotes the set of all solutions of the problem for (1.2). Here "dist" represents the usual distance set function. Under hypotheses (i) - (iv) one can use arguments very similar to those sketched above to establish that one has this method stability. If the sets Q^M are not defined through a mapping i^M as supposed above, one can still obtain this method stability if one replaces (iv) by the assumptions:

(v) If $\{q^M\}$ is <u>any</u> sequence with $q^M \in Q^M$, then there exists q^* in Q and subsequence $\{q^{M_k}\}$ with $q^{M_k} \to q^*$ in the $\tilde{Q}$ topology;

(vi) For <u>any</u> $q \in Q$, there exists a sequence $\{q^M\}$ with $q^M \in Q^M$ such that $q^M \to q$ in $\tilde{Q}$.

Similar ideas may be employed to discuss the question of <u>problem stability</u> for the problem of minimizing (1.1) over Q - i.e. the orginal problem and again compactness of the admissible parameter set plays a critical role. For discussions of other questions related to problem stability, see [19], [21] - and specifically Remark 5.1 of [21].

Compactness of parameter sets also appears to play an important role in computational considerations. For example, in certain problems the formulation outlined above (involving $Q^M = I^M(Q)$) results in a computational framework wherein the Q^M and Q all lie in some uniform set possessing compactness properties. The compactness criteria can then be reduced to uniform constraints on the derivatives of the admissible parameter functions. We have numerical examples which show that imposition of these constraints is necessary (and sufficient) for convergence of the resulting algorithms. (This offers a possible explanation for some of the numerical failures of such methods reported in the engineering literature -e.g. see [37].).

Thus we have that compactness of admissible parameter sets play a fundamental role in a number of aspects - both theoretical and computational- in parameter estimation problems. This compactness may be assumed (and imposed) explicitly as we have outlined here, or it may be included implicitly in the problem formulation through Tychonov regularization as recently discussed by Kravaris and Seinfeld [25]. In the regularization approach one restricts consideration to a subset Q_1 of parameters which has compact imbedding in Q, modifies the least-squares criterion to include a term which insures that minimizing sequences will be Q_1 bounded and hence compact in the original parameter set Q.

Having made a case for the role that compactness of admissible parameter sets might play in parameter estimation problems, we turn finally to the (not unrelated) focus of this note. In particular, we wish to discuss some problems in which a variational formulation (as opposed to the semigroup approximation framework we have used in many of our previous discussions of these problems - see [3,4,5,7,12,13,17]) permits relaxation of the compactness criteria needed in convergence, stability and/or computational analyses. We present several problems for which the variational framework can be used to give convergence arguments in the spirit of techniques commonly used in the finite-element approach (see [22] and the references therein) to initial-boundary value problems for partial differential equations. As we shall see below, the "energy functionals" in our case are parameter dependent and the arguments can become somewhat tedious in some instances.

In the next two sections we discuss problems for which the variational approach offers an alternative to the semigroup formulation. However, there are some problems for which the semigroup approach is not readily employed but for which a variational framework is rather natural. We present two su^h examples in Sections 4 and 5.

To facilitate our discussions, in some cases we restrict our remarks to problems in which we minimize J and J^N of (1.1) and (1.2) over a fixed set Q, relegating the role that approximating sets Q^M play to comments and referring the reader to [8] for an explanation of how one readily extends the ideas to problems of minimizing J^N over Q^M where Q^M approximates Q.

II. A "1-D Seismic" Inverse Problem

We consider the system

$$(2.1)\qquad q_1(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(q_2(x)\frac{\partial u}{\partial x}\right) \qquad t > 0,\ x \in \Omega = (0,1)$$

$$(2.2)\qquad \frac{\partial u}{\partial x}(t,0) + q_3 u(t,0) = f(t)$$

$$(2.3)\qquad \frac{\partial u}{\partial t}(t,1) + q_4\frac{\partial u}{\partial x}(t,1) = 0$$

$$(2.4)\qquad u(0,\cdot) = u_0\ ,\ u_t(0,\cdot) = v_0$$

and the associated inverse problem of estimating q_1, q_2, q_3, q_4, f, given a set of observations $\{y_{ij}\}$ for $\{u(t_i,x_j)\}$.

Such problems are motivated by certain versions of the so-called "1-D Seismic Inversion Problem" (see, e.g. [2], [18]). Roughly speaking, one has an elastic medium (e.g., the earth) with density q_1 and elastic modulus q_2. A perturbation of the system (explosions, or vibrating loads from specially designed trucks) near the surface ($x=0$) produces a source f for particle disturbances u that travel as elastic waves, being partially reflected due to the inhomogeneous nature of the medium. An important but difficult problem involves using the observed disturbances at the surface or at points along a "bore hole" to determine properties (represented by parameters in the system) of the medium. In the highly idealized 1-D "surface seismic" problem, one assumes that data are collected at the same point ($x=0$) where the original disturbance or "source" is located. In addition to this hypothesis, other unrealistic special assumptions are made about the nature of the traveling and reflected waves. Although the standard 1-D formulations are far from reality, exploration seismologists have developed techniques for processing actual field data (performing a series of experiments and "stacking" the data) so that the 1-D problems are generally accepted as useful and worthy subjects of investigation. Consequently, numerous papers (for some interesting references, see the bibliographies of [2], [18]) on the 1-D problems can be found in the research literature.

In many formulations of the seismic inverse problem, the medium is assumed to be the half-line $x > 0$ (with $x = 0$ the surface) while in others (especially some of those dealing with computational schemes) one finds the assumption of an artificial finite boundary (say at $x = 1$) at which no downgoing waves are reflected (an "absorbing" boundary). For the 1-D formulation this condition is embodied in a simple boundary condition of the form (2.3); here $q_4 \simeq \sqrt{q_2(1)/q_1(1)}$ and one can view this boundary condition as resulting from factoring the wave equation (2.1) at $x = 1$ and imposing the condition of "no upgoing waves" at $x = 1$.

Equation (2.1) is a 1-D version of the equations for an isotropic elastic medium while (2.2) represents an elastic boundary condition at the surface $x = 0$ (q_3 represents an elastic modulus for the restoring force produced by the medium).

In the usual seismic experiment, the medium is assumed initially at rest so that $u_0 = v_0 = 0$. While our analysis below can, with some tedium, be extended to treat the case of parameter dependent initial conditions, we shall assume that u_0, v_0 are given known functions.

For our discussions here, we shall also assume that the source term f is to be estimated in some function space although frequently this term can be parameterized in terms of a Euclidean set of parameters, thus simplifying somewhat the analysis. We shall also assume that $q_1(x) \equiv 1$ to facilitate our arguments (otherwise the analysis is somewhat more tedious and involves the use of parameter dependent inner products).

We reformulate the system (2.1)-(2.4) in variational or weak form, seeking a solution $t \to u(t)$ on $0 < t \leq T$, with $u(t) \in H^1(\Omega)$, satisfying

$$(2.5) \quad \langle u_{tt}, \psi \rangle + \langle q_2 Du, D\psi \rangle + \frac{q_2(1)}{q_4} u_t(t,1)\psi(1) - q_2(0)\left[q_3 u(t,0) - f(t)\right]\psi(0) = 0$$

for all $\psi \in H^1(\Omega)$, along with initial conditions

$$(2.6) \qquad u(0) = u_0 \ , \ u_t(0) = v_0 \ .$$

Here and throughout, unless otherwise noted, $\langle \ , \ \rangle$ denotes the usual inner product in $H^0 = L^2$ and $D = \frac{\partial}{\partial x}$. The parameters $q = (q_2, q_3, q_4, f)$ are assumed to be in some subset of $C(\Omega) \times R^1 \times R^1 \times C(0,T)$, although as we shall point out later, these smoothness requirements can be relaxed.

The system (2.5) - (2.6) can, for the purposes of analysis, be abstractly formulated using the state space $H = R^2 \times H^0(\Omega)$ for states $\hat{u}(t) = (u(t,0), u(t,1), u(t,\cdot))$. To be precise, define

$$V = \left\{(\eta,\xi,\psi) \in H : \eta = \psi(0),\ \xi = \psi(1),\ \psi \in H^1(\Omega)\right\}$$

and, for $\hat{v} = (\eta,\xi,v)$ in H, the operators $M_0\colon H \to H$, $N_0\colon H \to H$ by

$$M_0\hat{v} = (0,0,v) \quad N_0\hat{v} = (0, \frac{q_2(1)}{q_4}\xi,0).$$

We also define the functionals $a\colon V \times V \to R^1$, $b(t)\colon V \to R^1$ by

$$a(\hat{v}, \hat{\psi}) = \langle q_2Dv,D\psi\rangle - q_2(0)q_3v(0)\psi(0)$$

$$b(t)\hat{\psi} = q_2(0)f(t)\psi(0)\ .$$

Then we can rewrite (2.5) - (2.6) as

$$\langle M_0\hat{u}_{tt}, \hat{\psi}\rangle_H + \langle N_0\hat{u}_t,\hat{\psi}\rangle_H + a(\hat{u}(t),\hat{\psi}) + b(t)\hat{\psi} = 0 \tag{2.7}$$

where $\hat{u}(t), \hat{\psi} \in V$. We note that in this case the operators N_0, $b(t)$ and the functional a each depend on unknown parameters.

Standard arguments [28, p.273] can be used to guarantee existence of solutions $\hat{u}$ to (2.7), (2.6) satisfying $\hat{u} \in C(0,T;V)$, $\hat{u}_t \in C(0,T;H)$, $\hat{u}_{tt} \in H^0(0,T;V')$ with (2.7) being satisfied in a weak or distributional sense. Furthermore, one can rewrite (2.7) as a first order system and use semigroup techniques to argue that, under additional smoothness assumptions on the parameters and initial data, one obtains strong solutions that enjoy additional smoothness properties. We shall return to these considerations below.

Turning to approximation and convergence arguments, we shall work with our system in the form (2.5), (2.6) although we could equivalently use (2.7) in our considerations. We consider Galerkin approximations on finite dimensional subspaces $H^N \subset H^1(\Omega)$, $N = 1,2,\ldots$, and make the standing assumption on the orthogonal projections $P^N: H^0(\Omega) \to H^N$.

Assumption A: If $\Phi \in H^0(\Omega)$, then $P^N\Phi \to \Phi$ in H^0. For each $\Phi \in H^1(\Omega)$, we have $P^N\Phi \to \Phi$ in H^1.

If the observations for (2.1) - (2.4) for use in the least-squares functional are given in pointwise form $u(t_i,x_j)$ for the displacement or in an H^0 sense $u_x(t_i,\cdot)$ for the strain, it suffices for the convergence and stability arguments to argue that $u^N(t;q^N) \to u(t;q)$ in $H^1(\Omega)$ whenever $q^N \to q$ in an appropriate sense, where u^N is the Galerkin approximation to the original system (2.5), (2.6).

For parameters $q^N = (q_2^N, q_3^N, q_4^N, f^N)$ in an admissible parameter set Q, the approximating systems are given by: Find $u^N(t) \in H^N$ satisfying for all $\psi \in H^N$

(2.8) $$\langle u_{tt}^N,\psi\rangle + \langle q_2^N\, Du^N, D\psi\rangle + \frac{q_2^N(1)}{q_4^N} u_t^N(t,1)\psi(1) - q_2^N(0)\left[q_3^N u^N(t,0) - f^N(t)\right]\psi(0)=0,$$

(2.9) $$u^N(0) = P^N u_0$$
$$u_t^N(0) = P^N v_0 \; .$$

Regarding the admissible parameter set Q we make the standing assumptions:

<u>Assumption B:</u> The set Q is compact in the $\tilde{Q} = C(\Omega) \times R^1 \times R^1 \times H^1(0,T)$ topology and is contained in the set

$$\left\{(q_2, q_3, q_4, f) \in \tilde{Q} \mid q_2(x) \geq \nu > 0,\ q_3 < -\eta < 0,\ 0 < \beta \leq q_4 \leq \alpha\right\}$$

for some fixed positive constants α,β,ν,η.

Suppose then that $q^N \to \overline{q}$ in Q, where $\{q^N\}$ is <u>any</u> convergent sequence in Q, and let $u^N(q^N)$, $u(\overline{q})$ denote the corresponding solutions to (2.8), (2.5) respectively. Under Assumption A, we see from the inequality

$$\left|u^N(q^N) - u(\overline{q})\right|_1 \leq \left|u^N(q^N) - P^N u(\overline{q})\right|_1 + \left|P^N u(\overline{q}) - u(\overline{q})\right|_1$$

that it suffices to consider $z^N(t) \equiv u^N(t;q^N) - P^N u(t;\overline{q})$ and argue the convergence $z^N(t) \to 0$ in $H^1(\Omega)$ for each t in $[0,T]$.

Defining the "potential energy" functional $\Phi : H^1 \times H^1 \to R^1$ by

$$\Phi(q)(\phi,\psi) \equiv \langle q_2 D\phi, D\psi\rangle - q_2(0)q_3\phi(0)\psi(0) \tag{2.10}$$

and the "boundary damping" functional $B : H^1 \times H^1 \to R^1$ by

$$B(q)(\phi,\psi) \equiv \frac{q_2(1)}{q_4}\phi(1)\psi(1) \,, \tag{2.11}$$

we may use (2.5) (with $q = \overline{q}$) and (2.8) in

$$\langle z^N_{tt},\psi\rangle = \langle (u^N - u + u - P^N u)_{tt},\psi\rangle$$

to obtain

$$\langle z^N_{tt},\psi\rangle + \Phi(q^N)(z^N,\psi) + B(q^N)(z^N_t,\psi)$$

$$= \Phi(\overline{q})(u,\psi) + B(\overline{q})(u_t,\psi) - \Phi(q^N)(P^N u,\psi) - B(q^N)(P^N u_t,\psi) \tag{2.12}$$

$$+ \langle (I - P^N)u_{tt},\psi\rangle + \left[\overline{q}_2(0)\overline{f}(t) - q_2^N(0)f^N(t)\right]\psi(0)$$

for all $\psi \in H^N$. In addition to this equation, z^N satisfies the initial conditions (see (2.6) and (2.9))

$$z^N(0) = 0 \ , \ z_t^N(0) = 0. \tag{2.13}$$

Choosing $\psi = z_t^N$ (which is in H^N) in (2.12) and defining

$$\Delta\Phi(u) \equiv \Phi(\overline{q})(u,z^N) - \Phi(q^N)(P^N u,z^N) \tag{2.14}$$

$$\delta_3^N(t) \equiv \overline{q}_2(0)\overline{f}(t) - q_2^N(0)f^N(t) \quad , \tag{2.15}$$

we obtain from (2.12) the equation

$$\frac{1}{2}\frac{d}{dt}\left\{|z_t^N|^2 + \Phi(q^N)(z^N,z^N)\right\} + B(q^N)(z_t^N,z_t^N)$$

$$= \frac{d}{dt}\left\{\Delta\Phi(u)\right\} - \Delta\Phi(u_t) + \delta_3^N(t)\, z_t^N(t,0) \tag{2.16}$$

$$+ B(\overline{q})\,(u_t,z_t^N) - B(q^N)(P^N u_t,z_t^N) + \langle (I-P^N)u_{tt},z_t^N\rangle.$$

If we further define the total energy functional

$$E(q^N)(z^N) \equiv \left|z_t^N\right|^2 + \Phi(q^N)(z^N,z^N)$$

and the auxilary expressions (for notational convenience)

$$\delta_1^N(t) \equiv \frac{\bar{q}_2(1)}{q_4} u_t(t,1) - \frac{q_2^N(1)}{q_4^N} P^N u_t(t,1) \tag{2.17}$$

$$\delta_2^N(t) \equiv q_2^N(0) q_3^N P^N u(t,0) - \bar{q}_2(0)\bar{q}_3 u(t,0) \tag{2.18}$$

$$\Delta^N(t) \equiv \bar{q}_2 Du(t) - q_2^N DP^N u(t), \tag{2.19}$$

then we can rewrite (2.16) as

$$\frac{1}{2}\frac{d}{dt} E(q^N)(z^N(t)) + B(q^N)(z_t^N,z_t^N)$$

$$= \frac{d}{dt}\Delta\Phi(u) - \Delta\Phi(u_t) + \frac{d}{dt}\left\{\delta_3^N(t)\; z^N(t,0)\right\} - \delta_{3t}^N(t) z^N(t,0) \tag{2.20}$$

$$+ \delta_1^N(t) z_t^N(t,1) + \langle (I - P^N) u_{tt}, z_t^N \rangle.$$

We next observe that from Assumption B we have

$$B(q^N)(z_t^N, z_t^N) \geq \frac{\nu}{\alpha}\left|z_t^N(t,1)\right|^2$$

while the inequality $bc \leq \frac{1}{4\mu}b^2 + \mu c^2$ implies, with a proper choice of constants,

$$\delta_1^N(t)z_t^N(t,1) \leq \frac{\alpha}{4\nu}\left|\delta_1^N(t)\right|^2 + \frac{\nu}{\alpha}\left|z_t^N(t,1)\right|^2.$$

Using these inequalities in (2.20) we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}E(q^N)(z^N(t)) - \Delta\Phi(u) - \delta_3^N(t)z^N(t,0)\right\}$$

(2.21)

$$\leq -\Delta\Phi(u_t) + \frac{\alpha}{4\nu}\left|\delta_1^N(t)\right|^2 - \delta_{3t}^N(t)z^N(t,0) + \langle(I - P^N)u_{tt}, z_t^N\rangle.$$

Integrating this expression and using the facts that $\Delta\Phi(u)\big|_{t=0} = 0$, $z^N(0) \equiv 0$, and $E(q^N)(0) = 0$, we thus find

(2.22) $$E(q^N)(z^N(t)) \leq 2\Delta\Phi(u) + 2\delta_3^N(t)z^N(t,0) + 2\int_0^t G^N(t)dt$$

where G^N is defined as the function on the right side of the inequality (2.21). Finally, to make use of this bound on the parameter-dependent energy, we employ the following set of inequalities:

$$2\Delta\Phi(u)(t) \leq \frac{2}{\nu}\left|\Delta^N(t)\right|^2 + \frac{\nu}{2}\left|Dz^N(t)\right|^2 + \frac{4}{\nu\eta}\left|\delta_2^N(t)\right|^2 + \frac{\nu\eta}{4}\left|z^N(t,0)\right|^2,$$

$$2\delta_3^N(t)z^N(t,0) \leq \frac{4}{\nu\eta}\left|\delta_3^N(t)\right|^2 + \frac{\nu\eta}{4}\left|z^N(t,0)\right|^2 ,$$

$$-2\Delta\Phi(u_t)(\xi) \leq \frac{2}{\nu}\left|\Delta_t^N(\xi)\right|^2 + \frac{\nu}{2}\left|Dz^N(\xi)\right|^2 + \frac{4}{\nu\eta}\left|\delta_{2t}^N(\xi)\right|^2 + \frac{\nu\eta}{4}\left|z^N(\xi,0)\right|^2,$$

$$-2\delta_{3t}^N(\xi)z^N(\xi,0) \leq \frac{4}{\nu\eta}\left|\delta_{3t}^N(\xi)\right|^2 + \frac{\nu\eta}{4}\left|z^N(\xi,0)\right|^2,$$

$$2<(I - P^N)u_{tt}, z_t^N> \leq \left|(I - P^N)u_{tt}\right|^2 + \left|z_t^N\right|^2 ,$$

$$E(q^N)(z^N(t)) \geq \left|z_t^N(t)\right|^2 + \nu\left|Dz^N(t)\right|^2 + \nu\eta\left|z^N(t,0)\right|^2 .$$

Then (2.22) may be replaced by

$$\bar{E}(z^N(t)) \leq \Gamma^N(t) + \int_0^t \bar{E}(z^N(\xi))d\xi \tag{2.23}$$

where

$$\bar{E}(z^N(\xi)) \equiv \left|z_t^N(\xi)\right|^2 + \frac{\nu}{2}\left|Dz^N(\xi)\right|^2 + \frac{\nu\eta}{2}\left|z^N(\xi,0)\right|^2 \tag{2.24}$$

and

$$\Gamma^N(t) \equiv \frac{2}{\nu}\left|\Delta^N(t)\right|^2 + \frac{4}{\nu\eta}\left|\delta_2^N(t)\right|^2 + \frac{4}{\nu\eta}\left|\delta_3^N(t)\right|^2 + \int_0^T F(\xi)d\xi \tag{2.25}$$

with

$$F(\xi) \equiv \frac{2}{\nu}\left|\Delta_t^N(\xi)\right|^2 + \frac{\alpha}{2\nu}\left|\delta_1^N(\xi)\right|^2 + \frac{4}{\nu\eta}\left|\delta_{2t}^N(\xi)\right|^2 + \left|(I - P^N)u_{tt}(\xi)\right|^2 + \frac{4}{\nu\eta}\left|\delta_{3t}^N(\xi)\right|^2 . \tag{2.26}$$

Thus, from (2.23) and the Gronwall inequality, to establish that $\overline{E}(z^N(t)) \to 0$ (and hence that $z^N(t) \to 0$ in $H^1(\Omega)$), it suffices to argue that $\Gamma^N(t) \to 0$ for each t in $(0,T]$. In view of the definitions (2.15), (2.17) - (2.19), this convergence is easily argued under the Assumptions A, B (where $q^N \to \overline{q}$ in the $\tilde{Q}$ topology) and that $u(t) \in H^1(\Omega)$, $u_t(t) \in H^1(\Omega)$, $u_{tt} \in H^0([0,T] \times \Omega)$. To complete the discussions of this section, we shall comment on a number aspects of the above-outlined results in the form of several remarks.

<u>Remark 2.1</u>. Recalling Assumption B, we note that the formulation above requires the elastic moduli q_2 lie in $C(\Omega)$ and that the compactness properties of Q with respect to this component be in the $C(\Omega)$ sense. These are readily weakened by formulating the problem with the q_2 component replaced by $(q_2(0), q_2(1), q_2)$ in $R^1 \times R^1 \times L^\infty(\Omega)$ - see equation (2.5) - with a corresponding change in the $\tilde{Q}$ topology employed for the compactness statement in Assumption B. This is especially useful if one wishes to consider discontinuous elastic moduli (an important formulation in "multi-layered" seismic problems) and the ease with which such modifications are treated in the variational framework above make it ideally suited for convergence arguments when estimating discontinuous parameters.

Remark 2.2. In cases where the initial functions u_0, v_0 of (2.6) also depend on unknown parameters, i.e., $u_0 = u_0(q)$, $v_0 = v_0(q)$, then the initial conditions (2.13) must be replaced by $z^N(0) = P^N[u_0(q^N) - u_0(\overline{q})]$, $z_t^N(0) = P^N[v_0(q^N) - v_0(\overline{q})]$. The convergence arguments can be extended to this case if one makes appropriate smoothness assumptions on u_0, v_0 as functions of the parameters q.

Remark 2.3. Assumption A, the fundamental approximation hypotheses on H^N, is readily shown to hold if one chooses either the standard piecewise linear or cubic splines as approximation elements and hence these approximation schemes are included as special cases in the above treatment. The arguments and assumptions must be modified slightly if one wishes to include spectral families such as Legendre polynomials. (One obtains the convergence $P^N\Phi \to \Phi$ in H^1 for $\Phi \in H^{1+\epsilon}$ and Assumption A must be modified accordingly. This, in turn, requires that $u(t), u_t(t)$ be in $H^{1+\epsilon}$ in order to carry out the convergence arguments above.)

Remark 2.4. The presentation here assumes that one is performing the optimization in the least-squares fit-to-data over the admissible parameter set Q. In general, this is an infinite dimensional function space in the components q_2 and f. Thus one requires, as explained in Section 1, a second approximation family Q^M and a double limit procedure. Recalling our discussions from Section 1, we note that in the problems considered in this section it suffices to use the set $\tilde{Q} = C(\Omega) \times R^1 \times R^1 \times H^1(\Omega)$ in the compactness and approximation statements involving Q^M and Q.

Remark 2.5. In the presentation above, we found that to guarantee the desired convergence, it suffices to have $u(t)$ and $u_t(t)$ in $H^1(\Omega)$ and $u_{tt} \in H^0([0,T]x\Omega)$ where u is the solution of (2.5), (2.6) corresponding to any limit parameters $\bar{q}$ in Q. Without further smoothness assumptions on the problem data one cannot readily guarantee this desired regularity for u. For example, if $q_2 \in L^\infty(\Omega)$, $u_0 \in H^1(\Omega)$, $v_0 \in H^0(\Omega)$ and $f \in H^0(0,T)$, as we have already observed one can use variational theory to guarantee existence of weak solutions with $u_{tt} \in H^0(0,T; H^{-1})$. If, however, we assume $q_2 \in H^1$, $u_0 \in H^2$, $v_0 \in H^1$ and $f \in C^1$, then semigroup arguments similar to those given in [15] can be made to obtain strong solutions with $u(t) \in H^2$, $u_t(t) \in H^1$, and $u_{tt} \in H^0([0,T]x\Omega)$. Thus, q_2 in H^1 and sufficient smoothness on u_0, v_0, f will yield the desired smoothness for convergence.

Remark 2.6. The problem considered in this section was also investigated in [13] using a semigroup (Trotter-Kato approximation theorem) approach. Several differences in the results are noteworthy. First, we note that in the variational framework above, the approximating $(H^N \subset H^1(\Omega))$ basis elements are not required to satisfy parameter dependent boundary conditions (contrary to the situation in [13]). Furthermore, the unknown source term f in the boundary condition (2.2) can be treated directly here (in [13], the treatment required a transformation to a system with homogeneous boundary conditions and nonhomogeneous equation) and this relaxes the smoothness and compactness assumptions needed on the f component of Q. (A similar relaxation can be obtained in the framework of [13] using a slightly different method for transforming the nonhomogeneity in the boundary condition to the system equation.) In this regard we also note that the approach in [13] requires convergence $q_2^N \to q_2$ in $H^1(\Omega)$ as opposed to in $C(\Omega)$ (or $R^1 \times R^1 \times L^\infty(\Omega)$ - see Remark 2.1). Thus the related compactness assumption on the q_2 component Q_2 of Q is relaxed from H^1 to C compactness. This is potentially important in several respects. The characterization of compactness in $C(\Omega)$ is somewhat more natural (e.g. the Arzela-Ascoli lemma) than that in H^1 (e.g. see the embedding lemmas in [1]). Furthermore, in extending the ideas here or in [13] to treat estimation of discontinuous coefficients, compactness in H^1 (or a piecewise H^1 compactness) is more awkward and tedious to formulate than a concept of piecewise C or L^∞ compactness. Finally, we note that the mode of convergence required in the compactness of Q also dictates the type of approximating families Q^M one can use. We recall that the interpolation operators I^M (see [13] [32]) for both piecewise linear and cubic spline approximations satisfy $I^M(q) \to q$ uniformly in $q \in Q$ in either C or H^1 whenever $Q \subset H^2$. However there are occasion where one might desire to use the weaker convergence requirement (e.g. when dealing with discontinuous coefficients and a piecewise C topology), in which the variational formulation of the above presentation can prove advantageous.

III. Large Flexible Structures

In one important class of parameter estimation problems (see [5,20,24,29,35]), one wishes to estimate structural parameters (stiffness, damping, loading, etc.) in complex continuum models for elastic structures. The methods discussed in this paper can be successfully applied to such problems. To explain the approach, we consider a variable structure cantilevered damped beam with a tip body and base acceleration. Such a model might be used for example to describe the vibrations of shuttle attached payloads or large flexible spacecraft members. To be more specific we consider an Euler-Bernoulli beam of length ℓ with viscoelastic damping (a Kelvin-Voight solid) and tip mass m (instead of a tip body). Then the equations for transverse (planar) vibrations in the presence of an axial force due to base acceleration are given by

$$q_1 \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left\{q_2 \frac{\partial^2 u}{\partial x^2} + q_3 \frac{\partial^3 u}{\partial x^2 \partial t}\right\} = \frac{\partial}{\partial x}\left\{\sigma \frac{\partial u}{\partial x}\right\} + f,$$

$$t > 0,\ 0 < x < \ell,$$

$$\left[q_4 u_{tt} - \frac{\partial}{\partial x}\left\{q_2 \frac{\partial^2 u}{\partial x^2} + q_3 \frac{\partial^3 u}{\partial x^2 \partial t}\right\} + \sigma \frac{\partial u}{\partial x}\right]\Big|_{x=\ell} = g(t)$$

$$\left[q_2 \frac{\partial^2 u}{\partial x^2} + q_3 \frac{\partial^3 u}{\partial x^2 \partial t}\right]\Big|_{x=\ell} = 0$$

$$u(t,0) = \frac{\partial u}{\partial x}(t,0) = 0$$

$$u(0,\cdot) = \Phi\ ,\ u_t(0,\cdot) = \psi\ .$$

Here $q_1 = \rho(x)$ is the linear mass density, $q_2 = EI(x)$ is the flexural rigidity or stiffness, $q_3 = c_D I(x)$ is the viscoelastic damping coefficient, $q_4 = m$ is the tip mass, f is a distributed lateral load, and g is a transversely applied force on the tip mass. The internal tension σ due to axial loading from base acceleration is assumed to be a known function of $q_1 = \rho$, $q_4 = m$, and $q_5 = a_0$ = the base acceleration. From observations of the beam (displacement, velocity or strain), one might desire to estimate the parameters $q = (q_1, q_2, q_3, q_4, q_5) = (\rho, EI, c_D I, m, a_0)$ from a specified parameter set Q.

For such problems one can use a semigroup-Trotter-Kato approximation formulation to develop computational procedures - e.g. see [5,26,31]. However, some advantages are obtained in using a variational formulation (with "state" $\hat{u}(t) = (u(t,\cdot), u(t,\ell))$) similar to that in (2.5), (2.6) or (2.7). This is done in [6] and [15], [16] where detailed arguments for convergence are given. We shall not discuss them further here, except to note that the variational approach allows for a much weaker compactness criterion on the admissible parameter set Q. For example one can hypothesize compactness in the $C(0,\ell)$ norm (or in the $L^\infty(0,\ell)$ norm in a sense similar to that mentioned in Remark 2.1 above) with respect to the components representing EI and $c_D I$. (Compare this with the H^2 or H^2weak compactness assumptions in [5] and [26].)

IV. Bioturbation in Abyssal Sediments

In the previous two sections, we introduced problems (seismic and flexible structures) for which one can investigate parameter estimation techniques using either a semigroup formulation or a variational formulation. In this section and the next, we mention briefly two other classes of problems which are not readily treated with a semigroup formulation yet which can easily be analyzed in a variational setting. We first turn to problems related to the estimation of the effects of biological mixing in abyssal sediments.

Sediment formation in lakes and deep seas is of great importance to geophysical scientists who use core samples of this sediment in their investigations of the history of the earth. Unfortunately, the historical records contained in these core samples are often perturbed by a phenomenon called bioturbation [36] which is the mixing of sediments due to the activities of organisms near (on the order of 20-40 cm.) the sediment-water interface. These activities consist primarily of burrowing (e.g., for safety) and ingestion-excretion and are not easily described quantitatively.

An important goal of some geologists is to understand (quantitatively) bioturbation well enough so as to enable one to remove its effects and properly interpret the data in core samples, thereby sharpening the details in these geologic records. A number of increasingly sophisticated mathematical models have been proposed and a brief review of a number of these is given in [23]. One model of interest is the one proposed by Guinasso and Schink in [23].

Briefly, the model involves one-dimensional (depth) transport equations for a moving chamber (assumed uniform in horizontal directions) in which mixing and advective flow of material takes place. Depth in the chamber is represented by coordinates x, $0 \leq x \leq L$, and the chamber (and hence coordinate system) is assumed to be moving upward with a velocity $q_2 = q_2(t)$ [corresponding to sedimentation rate or build-up] so that it is always located in the top L cm. of the sediment, i.e., $x = 0$ is always at the water-sediment interface. The bottom of the chamber $x = L$ is located at that depth beyond which (it is assumed) no further changes (i.e. no bioturbation) in the historical records occur. If $u = u(t,x)$ is the concentration of material (e.g., shards of ash, tracer, etc.) with whose movement one is concerned, a model based on mass balance in the chamber, Fickian flux for the bioturbation, and appropriate boundary flux considerations is given by

$$(4.1) \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{q_1(x)\,\frac{\partial u}{\partial x}\right\} - q_2(t)\,\frac{\partial u}{\partial x} \qquad t > 0, \quad 0 < x < L\,,$$

$$(4.2) \qquad -q_1(0)\,\frac{\partial u}{\partial x}(t,0) + q_2(t)u(t,0) = 0$$

$$(4.3) \qquad -q_1(L)\,\frac{\partial u}{\partial x}(t,L) = 0\ .$$

Here q_1 is a depth dependent "bioturbation" coefficient.

To understand the effects of bioturbation on the distribution of material concentrations in core samples, it is sufficient then to know the parameters $q = (q_1, q_2, q_3)$, $q_3 = L$ and, of course, know that use of these parameters in the model gives one an accurate quantitative description of concentrations found in core samples. Given observations from core samples, this leads to a parameter estimation problem involving estimation of the functions q_1 and q_2 and the chamber length q_3. In [17], it is shown how to formulate and treat such problems in a discrete semigroup - Trotter-Kato approximation framework if one assumes that q_1 is chosen from a class of functions with finite dimensional parametric representation and q_2 is independent of time. If one wishes to treat more general problems of estimating $x \to q_1(x)$ and $t \to q_2(t)$ in general classes of functions, these are not so easily investigated using a semigroup setting (note that (4.2) involves time dependent unknowns). However, a variational formulation not unlike that given in [10] provides an amenable framework in which convergence arguments can be given under rather weak compactness assumptions on the admissible parameter sets. These arguments, to be given elsewhere, are similar in spirit to those given in [10] for transport problems involving estimation of spatially and temporally dependent coefficients.

V. Nonlinear Population Dispersal

In this section we turn to a brief discussion of estimation and a variational formulation for problems that are typical of population dispersal problems with transport coefficients (such as "diffusion" coefficients) that are density dependent. Nonlinearities of the general type we consider here are also important in porous media estimation problems.

Among the fundamental mechanisms often of interest to investigators in population dispersal (see [9,11,14,30,33,34]) are (in addition to the usual emigration-immigration, birth-death mechanisms): a dispersive mechanism associated with random movement or foraging; an attractive or repulsive force which induces directed movement of population members toward favorable or away from unfavorable environmental surroundings; and a mechanism representing population pressure due to interference between individuals in the population. In mathematical models for transport including such mechanisms, it is density dependent higher order terms that present difficulties in theoretical (and computational) considerations. To illustrate how a variational framework may be used for such problems, we shall sketch fundamental convergence arguments for problems involving estimation of the parameter function q in simple models of the form

$$(5.1) \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[q(t,x,u)\frac{\partial u}{\partial x} \right] \qquad t > 0,\ x \in \Omega = (0,1),$$

$$(5.2) \qquad u(t,0) = u(t,1) = 0$$

$$u(0,\cdot) = u_0. \tag{5.3}$$

It's not difficult to extend the ideas to many of the more detailed models (which include other desirable, but more easily treated mathematically, transport terms) studied in [9,14,30,33,34]. While we shall formulate the problem here in terms of estimating rather general "diffusion" coefficients q in (5.1) from a rather broadly defined class Q, we actually have as basic motivation the treatment of coefficients that are bounded below by density-independent base values, that are saturation limited with rates that are affine as a function of density in the range between the base-value and saturation thresholds. To be precise, our development has been motivated by problems where $\xi \to q(t,x,\xi)$ is continuous and has the form (see [14])

$$q(t,x,\xi) = \begin{cases} m(t,x) & \xi \le \zeta_0(t,x) \\ \alpha(t,x) + \beta(t,x)\xi & \zeta_0(t,x) \le \xi \le \zeta_1(t,x) \\ M(t,x) & \zeta_1(t,x) \le \xi . \end{cases} \tag{5.4}$$

In such problems we seek to estimate $\alpha,\beta,\zeta_0,\zeta_1$, (which determines q in (5.4) if the continuity assumption is invoked) from sets A,B,Γ_0,Γ_1 respectively. We shall sketch our ideas in terms of rather general conditions on the parameter set Q, noting that under appropriate assumptions on the sets A,B,Γ_0,Γ_1, the above example is included as a special case.

We first rewrite (5.1) - (5.3) in variational form, which consists of finding $u(t) \in H_0^1(\Omega)$ satisfying

$$(5.5) \qquad \langle u_t, \psi \rangle + \langle q(t,\cdot,u)Du, D\psi \rangle = 0$$

for all $\psi \in H_0^1(\Omega)$ along with initial conditions

$$(5.6) \qquad u(0) = u_0 .$$

To define an approximate state system, we assume we have chosen a family of finite-dimensional state spaces $H^N \subset H_0^1(\Omega)$ with orthogonal projections P^N : $H_0^1(\Omega) \to H^N$ in the $H^0(\Omega)$ inner product. We also assume Assumption A of Section 2 holds with H^1 replaced by H_0^1. The approximate systems are then given by seeking $u^N(t) \in H^N$ satisfying

$$(5.7) \qquad \langle u_t^N, \psi \rangle + \langle q^N(t,\cdot,u^N)Du^N, D\psi \rangle = 0$$

for all $\psi \in H^N$ and

$$(5.8) \qquad u^N(0) = P^N u_0 .$$

The parameters q^N are to be chosen from some admissible parameter set Q. In the usual manner (see Sections 1,2), for a convergence and stability analysis one desires to argue that $q^N \to \overline{q}$ in Q implies $u^N(q^N) \to u(\overline{q})$ for arbitrary sequences $\{q^N\}$ in Q. The mode of convergence in Q is, of course, also the sense in which we wish to define "compactness" of Q. In this particular example, we assume Q "compact" in the following sense:

Assumption C: Any sequence in Q has a convergent subsequence $\{q^k\}$ with limit q in Q in the sense

$$(5.9)\qquad \left|q^k(t,\cdot,v) - q(t,\cdot,v)\right|_\infty \to 0 \quad \text{in } H^0(0,T)$$

for every $v \in L^\infty(\Omega)$.

In the case where Q consists of functions of the form given in (5.4), it is straightforward to translate the compactness criteria of Assumption C into easily verifiable compactness criteria on the parameter sets A,B,Γ_0,Γ_1.

Further assumptions (again motivated by and easily verified for sets containing functions of the form (5.4)) on Q are necessary for convergence arguments and we therefore assume the following:

Assumption D: There is a constant $\nu > 0$ such that $\langle q(t,\cdot,v)D\psi,D\psi\rangle \geq \nu|\psi|_1^2$ for every $q \in Q$, $v,\psi \in H_0^1(\Omega)$.

Assumption E: There is a constant $M_0 > 0$ such that $|q(t,\cdot,v)\psi|_0 \leq M_0|\psi|_0$ for all $v \in L^\infty(\Omega)$, $\psi \in H^0(\Omega)$, $q \in Q$.

Assumption F: There exists $B \in L^{\infty}((0,T) \times \Omega)$ such that $|q(t,x,\xi) - q(t,x,\eta)| \leq B(t,x)|\xi - \eta|$ for all $\xi,\eta \in R^1$ and all $q \in Q$.

We note that Assumption F implies existence of a constant K such that $|q(t,\cdot,v) - q(t,\cdot,u)|_0 \leq K|v - u|_0$ for all $v,u \in H^0(\Omega)$.

We are now in a position to outline convergence arguments, for which it suffices (in the usual manner) to consider $z^N(t) \equiv u^N(t) - P^N u(t)$ where u^N, u, the solutions of (5.7), (5.8), (5.5), (5.6) corresponding to q^N, $\bar{q}$, respectively, where $q^N \to \bar{q}$ in the sense of (5.9) given in Assumption C and $\{q^N\}$ is any such sequence in Q.

From (5.5) - (5.8) we obtain $z^N(0) = 0$ and

$$(5.10) \qquad \langle z_t^N, \psi\rangle = \langle \bar{q}(t,\cdot,u)Du - q^N(t,\cdot,u^N)Du^N, D\psi\rangle + \langle (I - P^N)u_t, \psi\rangle$$

for all $\psi \in H^N$. Choosing $\psi = z^N$ in (5.10) and adopting the notation $\bar{q}(v)$, $q^N(v)$ for $\bar{q}(t,\cdot,v)$, $q^N(t,\cdot,v)$ throughout, we obtain

$$\frac{1}{2}\frac{d}{dt}|z^N|^2 = \langle \bar{q}(u)Du - q^N(u^N)Du^N, Dz^N\rangle + \langle (I - P^N)u_t, z^N\rangle,$$

or

$$(5.11) \qquad \frac{1}{2}\frac{d}{dt}|z^N|^2 + \langle q^N(u^N)Dz^N, Dz^N\rangle = \langle \bar{q}(u)Du - q^N(u^N)DP^N u, Dz^N\rangle + \langle (I - P^N)u_t, z^N\rangle.$$

From Assumption D we immediately find

$$(5.12)\quad \frac{1}{2}\frac{d}{dt}\left|z^N\right|^2 + \nu\left|z^N\right|_1^2 \leq \langle\bar{q}(u)Du - q^N(u^N)DP^Nu, Dz^N\rangle + \frac{1}{2}\left|(I - P^N)u_t\right|^2$$

$$+ \frac{1}{2}\left|z^N\right|^2.$$

Considering the first term on the right side of this inequality we find (all norms are the H^0 norm unless otherwise indicated)

$$(5.13)\quad \langle\bar{q}(u)Du - q^N(u^N)DP^Nu, Dz^N\rangle = \langle(\bar{q}(u) - q^N(u))Du, Dz^N\rangle$$

$$+ \langle(q^N(u) - q^N(u^N))Du, Dz^N\rangle + \langle q^N(u^N)D(u - P^Nu), Dz^N\rangle$$

$$\leq \frac{1}{\nu}\left|[\bar{q}(u) - q^N(u)]Du\right|^2 + \frac{\nu}{4}\left|Dz^N\right|^2$$

$$+ \frac{1}{\nu}\left|[q^N(u) - q^N(u^N)]Du\right|^2 + \frac{\nu}{4}\left|Dz^N\right|^2$$

$$+ \frac{1}{\nu}\left|q^N(u^N)D(u - P^Nu)\right|^2 + \frac{\nu}{4}\left|Dz^N\right|^2.$$

The terms in the right side of this inequality can be estimated (we use Assumptions E and F) as follows:

$$\left|q^N(u^N)D(u-P^Nu)\right| \leq M_0\left|D(u - P^Nu)\right|$$

$$\left|(q^N(u) - q^N(u^N))Du\right| \leq \left|Du\right|_\infty K\left|u - u^N\right| \leq \left|Du\right|_\infty K\left\{\left|(I - P^N)u\right| + \left|z^N\right|\right\}$$

$$\left|(\bar{q}(u) - q^N(u))Du\right| \leq \left|Du\right| \left|\bar{q}(u) - q^N(u)\right|_\infty .$$

Using these estimates and (5.13) in (5.12), we obtain

$$(5.14) \quad \frac{1}{2}\frac{d}{dt}\left|z^N\right|^2 + \frac{\nu}{4}\left|z^N\right|_1^2 \leq \mu\left|z^N\right|^2 + G^N(t)$$

where $\mu = \frac{1}{2} + \frac{2K^2}{\nu}\left|Du\right|_\infty^2$ and

$$G^N(t) = \frac{1}{2}\left|(I - P^N)u_t\right|^2 + \frac{M_0^2}{\nu}\left|D(u - P^N u)\right|^2 + \frac{2K^2}{\nu}\left|Du\right|_\infty^2\left|(I - P^N)u\right|^2$$

$$+ \frac{1}{\nu}\left|Du\right|^2\left|\bar{q}(u) - q^N(u)\right|_\infty^2 .$$

The remaining arguments are similar to those (integration, Gronwall, etc.) of Section 2. One easily obtains the convergence $z^N(t) \to 0$ in $H^0(\Omega)$ for each t and, actually, considering again (5.14), the additional results $z^N \to 0$ in $H^0(0,T; H^1(\Omega))$. Of course, appropriate smoothness (e.g. $u_t \in H^0([0,T] \times \Omega)$, $Du \in H^0(0,T; L^\infty(\Omega))$) assumptions on the solution u must be invoked.

We remark that the ideas sketched here readily extend to multi-dimensional domains Ω (see [10]) and that computational efforts based on the variational framework have in preliminary calculations (see [14]) been promising.

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